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Particle Creation in the Marginally Bound, Self Similar Collapse of Inhomogeneous Dust

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Abstract

We consider the evaporation of the (shell focusing) naked singularity formed during the self-similar collapse of a marginally bound inhomogeneous dust cloud, in the geometric optics approximation. We show that, neglecting the back reaction of the spacetime, the radiation on $\mathcal{I}^+$ tends to infinity as the Cauchy Horizon is approached. Two consequences can be expected from this result: (a) that the back reaction of spacetime will be large and eventually halt the formation of a naked singularity thus preserving the Cosmic Censorship Hypothesis and (b) mat-

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ter attempting to collapse into a naked singularity will radiate away energy at an intense rate, thereby possibly providing experimental signatures of quantum effects in curved spacetimes.

It is expected that very massive stars will undergo continual gravitational collapse. While the singularity theorems of Geroch, Hawking and Penrose establish that, under fairly general conditions, such a gravitational collapse will result in the formation of a singularity, they do not, by themselves, indicate whether the singularity will be hidden behind an event horizon or whether it will be visible to an outside observer. If the singularity is hidden the collapse ends in a black hole, and if it is visible, the collapse ends in a naked singularity. It is therefore an open problem in classical general relativity as to whether gravitational collapse ends in a black hole or in a naked singularity. Naked singularities have, so far, been considered undesirable and this has produced the so-called "cosmic censorship" hypothesis (CCH).[1] The CCH roughly states that the singularities arising in gravitational collapse of "reasonable" forms of matter with "reasonable" initial conditions are always hidden behind horizons and are not visible. Yet, attempts to prove this hypothesis on the classical level have been unsuccessful. On the contrary, studies of classical models of gravitational collapse have shown that both black holes and naked singularities can arise in collapse, depending on initial conditions.[2] If naked singularities do arise generically in collapse, they very possibly will have important observational astrophysical consequences.

Most studies of gravitational collapse assume spherical symmetry and even in this simplest of cases our understanding of the outcome of collapse is incomplete. The first model to be studied was the collapse of a homogeneous dust sphere (the Oppenheimer Snyder model).[3] This results in the formation of a black hole and most of our understanding of how a black hole forms in collapse, indeed, the censorship hypothesis itself, is motivated by this model. Later, the spherical collapse of inhomogeneous dust was investigated. This system is described by an exact solution of Einstein equations, which was given independently by Tolman and by Bondi.[4] The nature of the resulting singularities has been investigated by various authors[5] and it has been found that while some of the initial density and velocity distributions lead to black hole formation, other distributions result in the formation of naked singularities. There is a smooth transition from one phase to

the other, and the solution of Oppenheimer and Snyder is a very special case of a general inhomogeneous class.[6]

Another well-studied system is the spherical collapse of null dust, describing an exact solution of Einstein equations, the Vaidya spacetime. Once again, it has been found that both Black Holes and Naked Singularities result in this collapse, depending on the rate of infall of the null dust.[7]

Unfortunately, analyses of models of collapse with more realistic equations of state are hindered by the paucity of physically reasonable exact solutions. The collapse of a self-similar perfect fluid was investigated numerically by Ori and Piran[8] who found generic naked singularity solutions. These solutions, on their own merit, represent a serious violation of cosmic censorship. Further, numerical studies of collapsing scalar fields[9] indicate that the field disperses entirely without forming a singularity, in the domain of weak gravitational coupling. In the limit when the coupling is very strong, all the mass collapses to form a black hole, but, in the intermediate regime, part of the mass collapses to form a black hole and rest of it disperses. The transition point between dispersive and singular behavior is a naked singularity. The collapsed mass also shows a power-law dependence on the difference of the coupling parameter from its critical value, and the power-law index is independent of initial conditions. Similar behavior has also been found in the collapse of other forms of matter.[10]

It is likely of course that the initial conditions that lead to the classical collapse of matter into naked singularities are not acceptable quantum conditions. It is also likely that quantum effects, for example particle production in the presence of the strong gravitational fields involved toward the final stages, dominate the evolution at late times and prevent their formation. In this article we argue in favor of the latter. We will consider below a spherically symmetric model of inhomogeneous, marginally bound self similar dust collapse that arises as a special case of the general Tolman-Bondi collapse problem, analyze the causal structure of the spacetime and calculate the leading order contribution to the radiated power in the geomet-

ric optics approximation. We find that the radiated power diverges as the inverse square of the retarded distance from the Cauchy Horizon.

More specifically, the model is a solution of Einstein's equations with matter described by the stress energy tensor

$$T_{\mu\nu} \quad = \quad \epsilon(t,r)\delta^0_\mu\delta^0_\nu. \tag{1}$$

The metric is well known and given in comoving coordinates by

$$ds^2 \quad = \quad dt^2 \; - \; \tilde{R}'^2(t,r)dr^2 \; - \; \tilde{R}^2(t,r)d\Omega^2 \tag{2}$$

where the dust cloud is thought of as made up of concentric shells, each labeled by r. $\tilde{R}'(t,r)$ is the derivative of $\tilde{R}(t,r)$ with respect to r and $\tilde{R}(t,r)$ is the physical radius (the area of a shell labelled by r is $4\pi R^2(t,r)$) obeying, in the particular case of the marginally bound self similar collapse,

$$\tilde{R}(t,r) \quad = \quad r\left[1 \; - \; \frac{3\sqrt{\lambda}}{2}\frac{t}{r}\right]^{2/3}. \tag{3}$$

The physical radius is seen to depend on one parameter, λ, (the "mass parameter"). This parameter determines the total mass, $M(r)$, lying within the shell labeled by r as $2GM(r) = \lambda r$. The total mass of the dust is therefore $2GM = \kappa = \lambda r_o$ where r_o labels the outer boundary of the cloud. Now it can be shown that $\tilde{R}(t,r) = 0$ is a curvature singularity. This means that the singularity curves are $t_o(r) = 2r/(3\sqrt{\lambda})$, so that the last shell becomes singular at the time $t_o = 2/3\sqrt{r_o^3/\kappa}$.

Beyond $r = r_o$ one has the Schwarzschild spacetime

$$ds^2 \quad = \quad \left(1 - \frac{\kappa}{R}\right)dT^2 \; - \; \left(1 - \frac{\kappa}{R}\right)^{-1}dR^2 \; - \; R^2 d\Omega^2 \tag{4}$$

and the Tolman Bondi spacetime in (2) must be matched to (4) at the boundary. This means that the two metrics must agree on the three dimensional hypersurface

traced out by the outer boundary $r = r_o$. Comparing the angular coordinates, one concludes that

$$\tilde{R}(t, r_o) \quad = \quad R_o(t)$$

Therefore, requiring that the first fundamental forms agree on the hypersurface traced out by the collapsing outer boundary one has

$$\left(1 - \frac{\kappa}{\tilde{R}_o(t)}\right) \left[\frac{dT}{dt}\right]^2 \quad = \quad 1 \; + \; \left(1 - \frac{\kappa}{\tilde{R}_o(t)}\right)^{-1} \left[\frac{d\tilde{R}_o(t)}{dt}\right]^2 \tag{5}$$

where $\tilde{R}_o(t) = \tilde{R}(r_o, t)$. Using $\dot{\tilde{R}}(r_o, t) = -\sqrt{\kappa/\tilde{R}}$ one finds that (5) gives

$$T_o(t) \quad = \quad \int dt \left(1 - \frac{\kappa}{\tilde{R}_o(t)}\right)^{-1} \quad = \quad - \int d\tilde{R}_o(t) \sqrt{\frac{\tilde{R}_o(t)}{\kappa}} \left(1 - \frac{\kappa}{\tilde{R}_o(t)}\right)^{-1} \tag{6}$$

which integral may be solved to give

$$\begin{aligned} T_o(t) \quad &= \quad - \, 2\sqrt{\kappa \tilde{R}_o} \; - \; \frac{2}{3}\tilde{R}_o \sqrt{\frac{\tilde{R}_o}{\kappa}} \; + \; \kappa \ln \left| \frac{\sqrt{\tilde{R}_o} + \sqrt{\kappa}}{\sqrt{\tilde{R}_o} - \sqrt{\kappa}} \right| \\ &= \quad t \; - \; \frac{2}{3\sqrt{\kappa}} r_o^{3/2} \; - \; 2\sqrt{\kappa \tilde{R}_o} \; + \; \kappa \ln \left| \frac{\sqrt{\tilde{R}_o} + \sqrt{\kappa}}{\sqrt{\tilde{R}_o} - \sqrt{\kappa}} \right| \\ R_o(t) \quad &= \quad r_o \left[1 - a\frac{t}{r_o}\right]^{2/3} \end{aligned} \tag{7}$$

where we have set $a = 3\sqrt{\lambda}/2$. One can then show that the second fundamental forms agree by the relations in (7).

For the marginally bound, self-similar collapse under consideration, it is relatively simple to find null coordinates for this system. Consider the effective two

dimensional metric

$$ds^2 \;=\; dt^2 \;-\; \tilde{R}'^2(t,r)dr^2 \tag{8}$$

and change variables to z, x where $z \;=\; \ln r$, $x = t/r$. This gives

$$\begin{aligned} ds^2 \;&=\; r^2\left[dx^2 \;+\; 2xdxdz \;+\; (x^2 \;-\tilde{R}'^2(x))dz^2\right] \\ &=\; r^2(x^2 - \tilde{R}'^2)(d\tau^2 \;-\; d\chi^2) \end{aligned} \tag{9}$$

where

$$\begin{aligned} \tau \;&=\; z \;+\; \frac{1}{2}(I_- + I_+) \\ \chi \;&=\; \frac{1}{2}(I_- - I_+) \end{aligned} \tag{10}$$

in terms of

$$I_\pm(x) \;=\; \int \frac{dx}{x \pm \tilde{R}'} \tag{11}$$

We would like to choose null coordinates such that in the limit as $\lambda \to 0$ these reduce to the standard null coordinates in Minkowski space. Such coordinates are given by

$$\begin{aligned} u \;&=\; \begin{cases} +re^{I_-} & x - \tilde{R}' > 0 \\ -re^{I_-} & x - \tilde{R}' < 0 \end{cases} \\ v \;&=\; \begin{cases} +re^{I_+} & x + \tilde{R}' > 0 \\ -re^{I_+} & x + \tilde{R}' < 0 \end{cases} \end{aligned} \tag{12}$$

To further analyze the causal structure, it is now convenient to go over to the variable y defined by $y = \sqrt{\tilde{R}/r}$. In terms of y, the integrals $I_\pm$ can be written as

$$I_\pm \;=\; 9\int \frac{y^3dy}{3y^4 \;\mp\; ay^3 \;-\; 3y \;\mp\; 2a} \tag{13}$$

and the coordinates (12) become

$$\begin{aligned} u \;&=\; \begin{cases} +re^{I_-} & f_-(y) < 0 \\ -re^{I_-} & f_-(y) > 0 \end{cases} \\ v \;&=\; \begin{cases} +re^{I_+} & f_+(y) < 0 \\ -re^{I_+} & f_+(y) > 0 \end{cases} \end{aligned} \tag{14}$$

where

$$f_{\pm}(y) \quad = \quad 3y^4 \ \mp \ ay^3 \ - \ 3y \ \mp \ 2a. \tag{15}$$

Let $\alpha_i^{\pm}$ be the roots of $f_{\pm}(y)$, for $i \ \epsilon \ \{1, 2, 3, 4\}$. As $f_{\pm}$ are both real, they admit either 0, 2, or 4 real roots. The integrals can now be put in the form

$$I_{\pm} \quad = \quad 3 \int dy \left[\sum_{i=1}^{4} \frac{A_i^{\pm}}{(y - \alpha_i^{\pm})} \right] \tag{16}$$

where the $A_i^{\pm}$ are constants related to the coefficients of $f_{\pm}(y)$ and their roots by,

$$A_i^{\pm} \quad = \quad \frac{\alpha_i^{\pm 3}}{f'_{\pm}(\alpha_i^{\pm})} \tag{17}$$

In particular, the $A_i^{\pm}$ satisfy $\sum_i A_i^{\pm} = 1$. If all the roots are real, the solution is explicitly given by

$$\begin{aligned} u(y) \quad &= \quad \pm \, r \prod_{i=1}^{4} |y \ - \ \alpha_i^-|^{3A_i^-} \\ v(y) \quad &= \quad \pm \, r \prod_{i=1}^{4} |y \ - \ \alpha_i^+|^{3A_i^+} \end{aligned} \tag{18}$$

We will now consider the case in which there are two real roots and a conjugate pair of complex roots. As we will shortly show at least two real roots (possibly degenerate) are required for the existence of a globally naked singularity at the origin so we do not consider the case when all the roots are complex even though it may be carried out in the same spirit. Let us order the roots so that the first two, $\alpha_{1,2}$, are a complex conjugate pair and $\alpha_{3,4}$ are real. From (17) it follows that $A_{1,2}$ is also a complex pair whereas $A_{3,4}$ are real. Then the integrals are of the form

$$I \ = \ 3 \int dy \left[\sum_{i=1}^{4} \frac{A_i}{(y - \alpha_i)} \right] \ = \ 3 \left[A \ln(y - \alpha) \ + \ A^* \ln(y - \alpha^*) \ + \sum_{i=3,4} A_i \ln |y - \alpha_i| \right] \tag{19}$$

where α, α^* are the complex roots and A, A^* are the (complex) coefficients. Putting

$$A = |A|e^{i\phi}, \qquad y - \alpha = |y-\alpha|e^{i\xi} \tag{20}$$

so that the u, v coordinates have the explicit (and formal) solution

$$\begin{aligned} u(y) &= \pm r|y-\alpha^-|^{6|A^-|\cos\phi^-} e^{-6|A^-|\xi^- \sin\phi^-} \Pi_{i=3,4}|y-\alpha_i^-|^{3A_i^-} \\ v(y) &= \pm r|y-\alpha^+|^{6|A^+|\cos\phi^+} e^{-6|A^+|\xi^+ \sin\phi^+} \Pi_{i=3,4}|y-\alpha_i^+|^{3A_i^+} \end{aligned} \tag{21}$$

Consider the center ($r = 0$) at early times, $t < 0$. Then, because $y = (1 - at/r)^{1/3} \rightarrow \infty$, (18) gives (when all roots are real)

$$\begin{aligned} u &\rightarrow -r|y|^{3\sum_i A_i^-} = -r(1 - a\frac{t}{r}) \rightarrow at \\ v &\rightarrow -r|y|^{3\sum_i A_i^+} = -r(1 - a\frac{t}{r}) \rightarrow at \end{aligned} \tag{22}$$

This line is therefore given by $u = v$. When two of the roots are conjugate complex the line is still $u = v$ as we now show. Note that

$$\xi = \tan^{-1}\left(\frac{\mathrm{Im}(-\alpha)}{\mathrm{Re}(y-\alpha)}\right)$$

(y is real), so that as $y \rightarrow \infty$, $\xi \rightarrow 0$. Then clearly

$$\begin{aligned} u &\rightarrow -r|y|^{3(2\mathrm{Re}A^- + A_3^- + A_4^-)} \\ v &\rightarrow -r|y|^{3(2\mathrm{Re}A^+ + A_3^+ + A_4^+)} \end{aligned} \tag{23}$$

but since $\sum_i A_i^{\pm} = 1$, we have the same result as before.

The general solutions in equations (18, 21) are useful to analyze another limit, namely the singularity at $r \rightarrow at$. This means that $y \rightarrow 0$. Now when $y \rightarrow 0$,

$f_-(y) > 0$ and $f_+(y) < 0$. Then we see that (if all roots are real)

$$\begin{aligned} u &= -\, r \prod |\alpha_i^-|^{3A_i^-} \\ v &= r \prod |\alpha_i^+|^{3A_i^+} \end{aligned} \tag{24}$$

and, in particular,

$$\frac{v}{u} = -\, c = -\prod_i \frac{|\alpha_i^+|^{3A_i^+}}{|\alpha_i^-|^{3A_i^-}} \tag{25}$$

which is a negative constant, in general $\neq -1$. The singularity is therefore *spacelike* until the last shell, $r = r_o$, collapses at $t = t_o = r_o/a$. The case of a pair of conjugate complex roots trivially gives the same result. Beyond this point the singularity will be spacelike because it is just the Schwarzschild singularity in the exterior region. The behavior of the origin, $r = 0, t = 0$, is peculiar. It is the meeting point between two lines $u = v$ and $u = -cv$ and it's nakedness (coveredness) is far from clear. However, if a null ray originating at this point reaches the boundary at Kruskal coordinate $U < 0$ in the Schwarzschild region, it will reach $\mathcal{I}^+$ and then the origin will be globally naked.

We will be interested in the earliest null ray leaving the singularity and reaching $\mathcal{I}^+$ (the Cauchy Horizon) as well as the earliest null ray that strikes the singularity from $\mathcal{I}^-$. These rays can be expected to intersect the first singular shell at $r = 0, t = 0$, so it is natural to carefully examine the null rays passing through this point. The origin, being the intersection of the lines $u = v$ and $v = -cu$ ($c \neq 1$ in general), corresponds to the point $u = 0 = v$. Now any null ray traveling toward $\mathcal{I}^+$ with $u = 0$ must have either $r = 0$ or $I_- \to -\infty$. Therefore, when $r \neq 0$, such a ray is possible if and only if $y = \alpha_k^-$ for some *real* root, α_k^- of the polynomial $f_-(y)$. Indeed such a root may not exist, in which case the singularity is not naked as no null rays can emanate from it. If a real root exists however, at least one null ray leaves this point and reaches the boundary. The existence of real roots of the polynomials $f_-(y)$ is therefore a necessary condition for the nakedness of the origin. This places a constraint on the possible value of the constant a in the mass

function. One finds that real roots exist provided that $a < a_c \sim 0.638$.[11] Each root corresponds to a null ray emanating from $u = 0 = v$ and there are at least two of them, if any at all. Because $y = \alpha_i$ implies that $t = r(1 - \alpha_i^3)/a$, we choose the largest real root of $f_-(y)$ as the one that gives the earliest null ray emanating from $u = 0 = v$ and call it α_-. Thus, $y = \alpha_-$ is the Cauchy horizon.

A similar reasoning can now be given for the incoming rays passing through $u = 0 = v$. Again any ray with $v = 0$ for $r \neq 0$ must have $I_+ \to -\infty$, which is possible only if $y = \alpha_k^+$ for some *real* root, α_k^+ of the polynomial $f_+(y)$. Now, $f_+(y)$ admits two real roots, one unphysical (negative) and one positive. Again, call the (positive) physical root α_+.

What we have described above is pictured in the Penrose diagram of figure I.

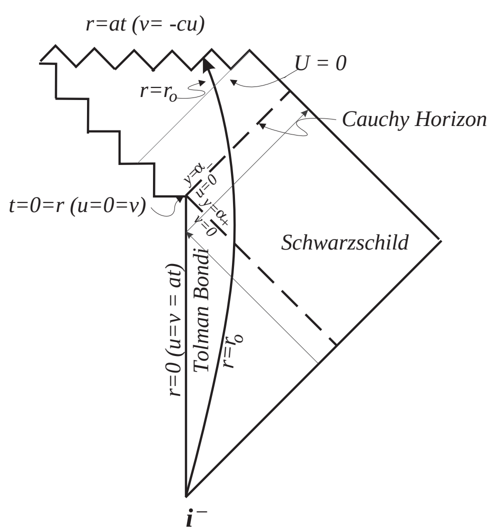

Figure I

We will henceforth consider rays in the neighborhood of the lines given by $y = \alpha_-$ for outgoing rays and $y = \alpha_+$ for incoming rays. The precise values of $\alpha_\pm$ in terms of the mass parameter will not interest us for this work but we will Taylor expand about these two values, considering $y_\pm = \tilde{y}_\pm + \alpha_\pm$.

Returning to (7), one can rewrite the Schwarzschild radial coordinate and time

on the boundary as follows

$$\begin{aligned} R_o(y) &= r_o y^2 \\ T_o(y) &= -\frac{r_o}{a}y^3 - \frac{4}{3}ar_o y - \frac{4}{9}a^2 r_o \ln\left|\frac{3y/2a-1}{3y/2a+1}\right| \end{aligned} \tag{26}$$

Therefore, the Eddington-Finkelstein null coordinates on the boundary, $\tilde{U}_o(y) = T_o(y) - R_{o*}(y)$, $\tilde{V}_o(y) = T_o(y) + R_{o*}(y)$, (where R_{o*} is the tortoise coordinate) take the form

$$\begin{aligned} \tilde{U}_o(y) &= -\frac{r_o}{a}y^3 - \frac{4}{3}ar_o y - r_o y^2 - \frac{8}{9}a^2 r_o \ln|3y/2a-1| \\ \tilde{V}_o(y) &= -\frac{r_o}{a}y^3 - \frac{4}{3}ar_o y + r_o y^2 + \frac{8}{9}a^2 r_o \ln|3y/2a+1| \end{aligned} \tag{27}$$

It is now clear that the earliest null outgoing ray, $u = 0$, from the origin (the Cauchy Horizon) within the cloud strikes the boundary at $y = \alpha_-$ and translates into the null outgoing ray

$$\tilde{U}_o^{(0)} = -\frac{r_o}{a}\alpha_-^3 - \frac{4}{3}ar_o\alpha_- - r_o\alpha_-^2 - \frac{8}{9}a^2 r_o \ln|3\alpha_-/2a-1| \tag{28}$$

which is never infinite ($2a/3$ is not a root of $f_-(y)$). This null ray corresponds to a finite value of $\tilde{U}$ and will therefore reach $\mathcal{I}^+$, so the existence of real roots of $f_-(y)$ turns out to be not just necessary, but a sufficient condition for the origin to be globally naked. The same argument applies to the infalling ray(s): the earliest null ray to pass through the origin is the ray corresponding to the value $y = \alpha_+$, or

$$\tilde{V}_o^{(0)} = -\frac{r_o}{a}\alpha_+^3 - \frac{4}{3}ar_o\alpha_+ + r_o\alpha_+^2 + \frac{8}{9}a^2 r_o \ln|3\alpha_+/2a+1| \tag{29}$$

and, again, since $-2a/3$ is not a root of $f_+(y)$, $\tilde{V}$ is not infinitely negative and such a ray will have come from $\mathcal{I}^-$. Thus, the existence of a positive real root of $f_+(y)$ is sufficient to ensure that at least one infalling ray from $\mathcal{I}^-$ will intersect the origin.

The next question we must address is the relationship between the $\tilde{U}, \tilde{V}$ coordinates in the exterior and the u, v coordinates (equations (18, 21)) on the boundary. This is difficult to do in general, but if we confine our study to rays that are "close" to $u = 0$ and $v = 0$ we can arrive at some conclusion regarding the quantum radiation on $\mathcal{I}^+$ near the Cauchy horizon. "Close" will be taken to mean linearizations about $y = \alpha_\pm$ respectively for incoming rays and outgoing rays.

First consider outgoing rays. For $y \sim \alpha_-$, define $y = \tilde{y} + \alpha_-$ and find that for small $\tilde{y}$

$$I_- \quad \sim \quad \gamma_- \ln \tilde{y} \quad + \quad \mathcal{O}(y) \tag{30}$$

where

$$\gamma_- \quad = \quad \frac{3\alpha_-^3}{f'_-(\alpha_-)} \tag{31}$$

giving

$$u \quad = \quad -r|\tilde{y}|^{\gamma_-} \quad \rightarrow \quad y - \alpha_- \quad = \quad \left(-\frac{u}{r}\right)^{1/\gamma_-} \tag{32}$$

Therefore in terms of u (on the boundary) we can write $\tilde{U}$ as follows

$$\tilde{U} \quad \sim \quad \tilde{U}^{(0)}(\alpha_-) \; + \; \Gamma_-(\alpha_-)(y - \alpha_-) \quad = \quad \tilde{U}^{(0)}(\alpha_-) \; + \; \Gamma_-(\alpha_-)\left(-\frac{u}{r_o}\right)^{1/\gamma_-} \tag{33}$$

where

$$\Gamma_- \quad = \quad -\, 9\frac{r_o\alpha_-^3}{a(3\alpha_- - 2a)} \quad < \quad 0 \;\; \text{when} \;\; a < a_c \tag{34}$$

Figure II is a plot of Γ_- as a function of a for $a < a_c$.

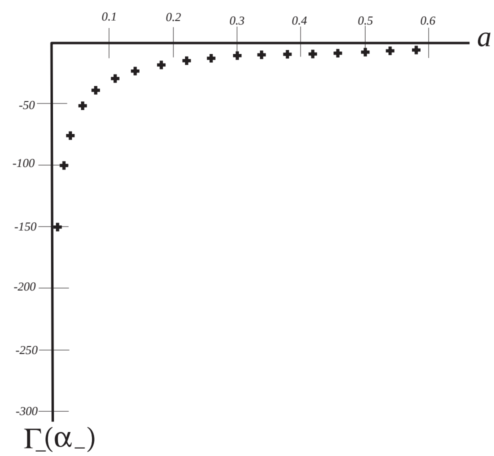

Figure II

Likewise, for incoming rays, put $y = \tilde{y} + \alpha_+$ and find that

$$I_+ \quad = \quad \gamma_+ \ln \tilde{y} \quad + \quad \mathcal{O}(y) \tag{35}$$

where

$$\gamma_+ \quad = \quad \frac{3\alpha_+^3}{f'_+(\alpha_+)} \tag{36}$$

giving

$$v \quad = \quad -r|\tilde{y}|^{\gamma_+} \quad \rightarrow \quad y - \alpha_+ \quad = \quad \left(-\frac{v}{r}\right)^{1/\gamma_+} \tag{37}$$

Thus, in terms of v (on the boundary) we can write $\tilde{V}$ as follows

$$\tilde{V} \quad \sim \quad \tilde{V}^{(0)}(\alpha_+) \ + \ \Gamma_+(\alpha_+)(y - \alpha_+) \quad = \quad \tilde{V}^{(0)}(\alpha_+) \ + \ \Gamma_+(\alpha_+)\left(-\frac{v}{r_o}\right)^{1/\gamma_+} \tag{38}$$

where

$$\Gamma_+ \quad = \quad -9\frac{r_o\alpha_+^3}{a(3\alpha_+ + 2a)} \quad < \quad 0 \ \ \text{when} \ \ a < a_c \tag{39}$$

Figure III is a plot of Γ_- as a function of a for $a < a_c$.

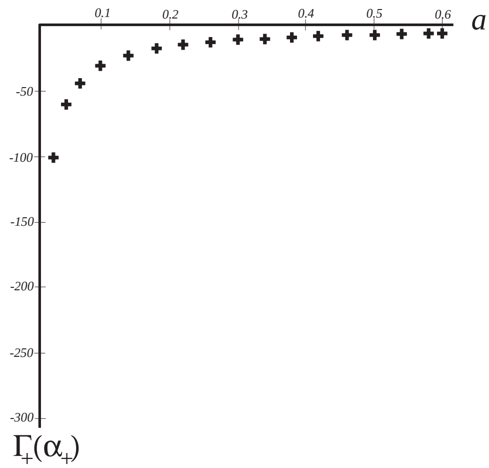

Figure III

We are now in a position to compute the radiated power close to the Cauchy horizon in the geometric optics approximation

The geometric optics approximation is a fairly general way to obtain the power radiated past $\mathcal{I}^+$ by the lower angular momentum modes in an asymptotically flat spacetime in which radial null rays define a one to one map between the $\mathcal{I}^-$ and $\mathcal{I}^+$. The method originated in the work of Moore[12], Hawking[13] and DeWitt[14], was later used by Fulling and Davies[15] in a two dimensional model examining the radiation from a moving mirror and by Ford and Parker[16] to study some four dimensional collapse models. In their study, Ford and Parker considered the collapse of a dust cloud leading to the formation of a shell crossing naked singularity and found that the energy flux of the created scalar particles remained finite up to the time of formation of the singularity. They also considered the collapse of charged shells (for which the charge exceeds the mass) leading to naked singularities and observed that, for these, the flux of created particles is infinite. However, naked singularities are formed in these models if the proper mass is negative or if Einstein's equations are not imposed. The model we are considering, on the other hand, is a genuine solution of Einstein's equations with reasonable matter, yet, as we show, that the result is the same.

Mode solutions of the free scalar equation $\nabla^2\phi = 0$ have the general solutions

$$f_{\omega lm} \sim (4\pi\omega)^{-1/2} \begin{cases} R^{-1} Y_{lm} e^{-i\omega\tilde{U}} \\ R^{-1} Y_{lm} e^{-i\omega\tilde{V}} \end{cases} \tag{40}$$

at spatial infinity, $R \to \infty$, where $\tilde{U}$ and $\tilde{V}$ are the null coordinates defined earlier. Consider an expansion of a massless scalar field, ϕ, in terms of a complete set of modes

$$\phi = \sum_{lm} \int d\omega [f_{\omega lm} a_{\omega lm} + f^*_{\omega lm} a^\dagger_{\omega lm}] \tag{41}$$

where the $f_{\omega lm}$ are appropriately normalized and reduce to the second of the set in (40) in the remote past. The " in" vacuum is then defined by $a_{\omega lm}|0\rangle = 0$ and corresponds to the absence of incoming radiation from $\mathcal{I}^-$.

One is interested in the form of $f_{\omega lm}$ in the remote future. An incoming ray, $\tilde{V} =$ const., originating at $\mathcal{I}^-$, will pass through the geometry of the spacetime to become an outgoing null ray that arrives at $\mathcal{I}^+$ at a time $\tilde{U} = \mathcal{F}(\tilde{V})$. Alternatively, a ray that arrived on $\mathcal{I}^+$ at $\tilde{U}$ will have originated on $\mathcal{I}^-$ at $\tilde{V} = \mathcal{G}(\tilde{U})$. Thus a wave packet formed from plane waves $e^{-i\omega\tilde{V}}$ becomes, at late times, an outgoing wave packet formed from plane waves $e^{-i\omega\mathcal{G}(\tilde{U})}$. It is necessary, therefore, to consider a solution of the massless wave equation which has the form

$$f_{\omega lm} = (4\pi\omega)^{-1/2}(e^{-i\omega\tilde{V}} + e^{-i\omega\mathcal{G}(\tilde{U})}) R^{-1} Y_{lm} \tag{42}$$

in the asymptotic region. These modes reduce to the standard (incoming) modes on $\mathcal{I}^-$ and have a complicated (outgoing) form on $\mathcal{I}^+$.

The flux of energy radiated to $\mathcal{I}^+$ by the massless scalar field particles is then given by the expectation value of the off-diagonal component of the stress energy tensor of the massless scalar field,

$$T^R_T = \frac{1}{2}\left\{\phi^{,R}, \phi_{,T}\right\}_+, \tag{43}$$

where $\{\}_+$ represents the anticommutator. The operator is naturally not well

defined but may be renormalized by point-splitting, giving

$$\langle 0|T_T^R|0\rangle \quad = \quad \frac{1}{4\pi R^2}\sum_{lm}|Y_{lm}|^2\left[\frac{1}{4}\left(\frac{\mathcal{G}''}{\mathcal{G}'}\right)^2 \quad - \quad \frac{1}{6}\frac{\mathcal{G}'''}{\mathcal{G}'}\right] \tag{44}$$

The total power radiated across a sphere or radius r at late times is therefore

$$P \quad = \quad \int\langle 0|T_T^R|0\rangle R^2\sin\theta d\theta d\phi \quad = \quad \frac{1}{4\pi}\sum_{l,m}\left[\frac{1}{4}\left(\frac{\mathcal{G}''}{\mathcal{G}'}\right)^2 \quad - \quad \frac{1}{6}\frac{\mathcal{G}'''}{\mathcal{G}'}\right] \tag{45}$$

where the sum is over all (angular momentum) modes, l, m, and is thus formally infinite. However, the geometric optics approximation is invalid for the higher angular momentum modes. This is due to the centrifugal effective potential that causes the mode function to scatter to infinity before it can pass through the region of high curvature. One expects that this effect will reduce the flux considerably for modes of large angular momentum, therefore the expression in (45) is expected to give a good approximation for small l but for large l the radiated power is expected to diminish rapidly, becoming effectively vanishing† [17]. One can write the same expression in terms of $\mathcal{F}$ as follows

$$P \quad = \quad \int\langle 0|T_T^R|0\rangle R^2\sin\theta d\theta d\phi \quad = \quad \frac{1}{24\pi}\sum_{l,m}\left[\frac{\mathcal{F}'''}{(\mathcal{F}')^3} \quad - \quad \frac{3}{2}\left(\frac{\mathcal{F}''}{\mathcal{F}'^2}\right)^2\right] \tag{46}$$

The function $\mathcal{F}(\tilde{V})$ is the result of tracing a null ray coming in at $\tilde{V} =$ const. from $\mathcal{I}^-$ traveling across the boundary, through the center and out across the boundary again to become the ray $\tilde{U} =$ const.$= \mathcal{F}(\tilde{V})$ on $\mathcal{I}^+$. The heart of the geometric optics approximation is therefore in the determination of the function $\mathcal{F}(\tilde{V})$. As we have set it up, this is now an easy task for the problem at hand.

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† We wish to thank S. Fulling for clarifying this point and for correcting certain bibliographical errors in the original version of this paper.

Therefore, consider a ray $\tilde{V}$ = const. in the infinite past. We are interested only in the region on $\mathcal{I}^+$ that is close to the Cauchy horizon, so the approximations in (33) and (38) will suffice. As the null ray crosses the boundary, we have

$$\tilde{V}(v) \;=\; \tilde{V}^{(0)} \;+\; \Gamma_+ \left(-\frac{v}{r_o}\right)^{\frac{1}{\gamma_+}} \tag{47}$$

This expression can be inverted to give

$$v(\tilde{V}) \;=\; -\, r_o \left[\frac{\tilde{V}^0 - \tilde{V}}{|\Gamma_+|}\right]^{\gamma_+} \tag{48}$$

where we have used the fact that Γ_+ is negative. Next, reflecting about the center (here, $u = v$) gives

$$u(\tilde{V}) \;=\; -\, r_o \left[\frac{\tilde{V}^0 - \tilde{V}}{|\Gamma_+|}\right]^{\gamma_+} \tag{49}$$

Now as the outgoing ray crosses the outer boundary, we have the relation

$$\begin{aligned} \tilde{U}(u) \;&=\; \tilde{U}^{(0)} \;-\; \Gamma_- \left(-\frac{u}{r_o}\right)^{\frac{1}{\gamma_-}} \\ \rightarrow \;\; \tilde{U}(\tilde{V}) \;&=\; \tilde{U}^{(0)} \;-\; |\Gamma_-| \left[\frac{\tilde{V}^0 - \tilde{V}}{|\Gamma_+|}\right]^{\frac{\gamma_+}{\gamma_-}} \end{aligned} \tag{50}$$

where now we have used the fact that Γ_- is negative. Thus, the right hand side of (50) is $\mathcal{F}(\tilde{V})$ and it has the form

$$\mathcal{F}(\tilde{V}) \;=\; A \;-\; B(\tilde{V}^{(0)} - \tilde{V})^{\frac{\gamma_+}{\gamma_-}} \tag{51}$$

where B is a positive constant which is given in terms of the roots, $\alpha_\pm$ given before. We can now write down the power radiated as a function of $\tilde{V}$. ¿From

(46), it follows that

$$P(\tilde{V}) \quad = \quad \frac{1}{48\pi B^2}\left[\frac{1-\gamma^2}{\gamma^2(\tilde{V}^{(0)}-\tilde{V})^{2\gamma}}\right] \qquad \gamma \neq 0 \tag{52}$$

where

$$\gamma \quad = \quad \frac{\gamma_+}{\gamma_-} \tag{53}$$

The expression can also be expressed as a function of the outgoing null coordinate $\tilde{U}$ on $\mathcal{I}^+$ by simply exchanging $\tilde{V}$ for $\tilde{U}$ in (52), using (50), or directly computing the radiated power from (45). The result is

$$P(\tilde{U}) \quad = \quad \frac{1}{48\pi}\left[\frac{1-\gamma^2}{\gamma^2(\tilde{U}^{(0)}-\tilde{U})^2}\right] \tag{54}$$

Now, $0 < \gamma \leq 1$ for all a in the range that admits naked singularities (as shown in figure IV), approaching unity in the limit $a \to 0$ and decreasing to a minimum of ~ 0.6. It is imaginary when $a > a_c$, which is to be expected because there is no outgoing ray from the origin and the entire treatment breaks down. One sees clearly that the flux diverges as the Cauchy horizon is approached.

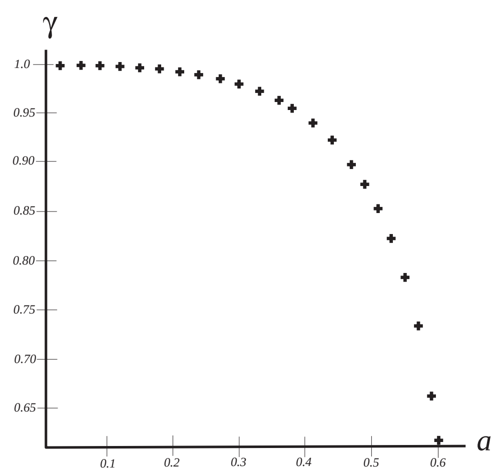

Figure IV

In this article we have examined scalar particle production in the neighborhood of the Cauchy horizon of the shell focusing naked singularity formed by the self

similar collapse of an inhomogeneous, marginally bound, dust cloud and found that the radiated power will increase rapidly as the Cauchy horizon is approached. Particle production is a purely kinematical phenomenon and not directly related to the Einstein equations. The phenomenon we have described should not depend, therefore, on the particular solution that has been examined. On the contrary, we expect this to occur generically when regions of high curvature are exposed to the asymptotic observer.

What consequences could this have? We imagine a dust cloud whose initial conditions are such as to lead to the formation of a naked singularity. As the successive shells begin to form the singularity, the uncertainty principle takes control leading to a steadily growing and finally intense outgoing radiation of energy from the cloud as the Cauchy horizon is approached. In the final stages of collapse, this radiation should occur at extremely high energies that will possibly be visible to the asymptotic observer and will provide signatures of the behavior of quantum fields in curved spacetimes. Owing to the strong back reaction of the spacetime in the final stages, we do not expect that the naked singularity will actually form. Indeed, the CCH may have it's origins in precisely such an effect.

It may be argued that the correct arena in which such strong gravitational fields and the back reaction of spacetime should be studied is string theory, as string theory provides a consistent quantum theory of gravity. The outcome suggested above has been verified in some two dimensional models of string gravity.[18] We believe it is of interest to pursue this investigation, both from a theoretical standpoint as well as an experimental.

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Figure Captions:

I. Penrose diagram for the spacetime of the marginally bound, self-similar collapse of inhomogeneous dust. The heavy dashed outgoing null ray is the Cauchy horizon at $y = \alpha_-$ and the lighter null lines trace a ray that originated at some advanced time $\tilde{V}$ on $\mathcal{I}^-$ and crosses $\mathcal{I}^+$ close to the Cauchy horizon.

II. The behavior of $\Gamma_-(\alpha_-)$ as a function of the mass parameter a and for values of a that admit naked singularity solutions.

III. The behavior of $\Gamma_+(\alpha_+)$ as a function of the mass parameter a and for values of a that admit naked singularity solutions.

IV. The behavior of the exponent $\gamma = \gamma_+/\gamma_-$ in (53) as a function of the mass parameter a and for values of a that admit naked singularity solutions.

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